

# An Ergodic Result

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## Abstract

A rather general ergodic type scheme is presented on arbitrary sets  $X$ , as they are generated by arbitrary mappings  $T : X \longrightarrow X$ . The structures considered on  $X$  are given by suitable subsets of the set of all of its finite partitions. Ergodicity is studied not with respect to subsets of  $X$ , but with the *inverse limits* of families of finite partitions.

## 1. The Setup

Let  $(X, \Sigma, T)$  be as follows :  $X$  is an arbitrary nonvoid set,  $\Sigma \subseteq \mathcal{P}(X)$  is a nonvoid set of subsets of  $X$ , while  $T : X \longrightarrow X$ .

The issue considered, as usual in Ergodic Theory, is the behaviour of the sequence of iterates  $T^n(x)$ ,  $n \in \mathbb{N}_+ = \{1, 2, 3, \dots\}$ , for an arbitrary given  $x \in X$ . Of a main interest in this regard is of course the case when  $X$  is infinite.

A simplest and natural way to follow is to consider a *partition* of  $X$ , and see how the mentioned sequence of iterates may possibly move through the various sets of that partition. In this regard, a further simplest and natural case is when the partitions considered for  $X$  are *finite*, and thus at least one of their sets must contain *infinitely* many terms of any such sequence of iterates.

As it turns out, a number of properties can be obtained simply form

the finite versus infinite *interplay* as set up above, an interplay slightly extending the usual pigeon-hole principle. However, in order to obtain such properties, one may have to *shift* the usual focus which tends to be concerned with the relationship between the mapping  $T$  and its iterates  $T^n$ , with  $n \in \mathbb{N}_+$ , and on the other hand, the various subsets  $A \subseteq X$ . Namely, this time one is dealing with the relationship between the mapping  $T$  and its iterates  $T^n$ , with  $n \in \mathbb{N}_+$ , and on the other hand, whole *families* of *finite* partitions  $\Delta$  of  $X$ .

Let us therefore consider

$$(1.1) \quad \mathcal{FP}(X, \Sigma)$$

the set of all *finite* partitions  $\Delta$  of  $X$  with nonvoid subsets in  $\Sigma$ , thus  $\Delta \subseteq \Sigma$ ,  $\Delta$  is finite, and  $X = \bigcup_{A \in \Delta} A$ , where for  $A \in \Delta$  we have  $A \neq \phi$ , however in general, none of  $A \in \Delta$  need to be finite.

Given  $x \in X$  and  $\Delta \in \mathcal{FP}(X, \Sigma)$ , then obviously

$$(1.2) \quad \exists A \in \Delta : \{ n \in \mathbb{N}_+ \mid T^n(x) \in A \} \text{ is infinite}$$

since  $\Delta$  is finite.

Let us therefore denote

$$(1.3) \quad \Delta(x) = \{ A \in \Delta \mid \{ n \in \mathbb{N}_+ \mid T^n(x) \in A \} \text{ is infinite} \}$$

and then (1.2) implies

$$(1.4) \quad \Delta(x) \neq \phi$$

### Problem 1

Given  $x \in X$ , what happens with  $\Delta(x)$ , when  $\Delta$  ranges over  $\mathcal{FP}(X, \Sigma)$  ?

### Example 1

Let  $X = \mathbb{N}$ ,  $\Sigma = \mathcal{P}(\mathbb{N})$  and consider the following three cases of mappings  $T : \mathbb{N} \longrightarrow \mathbb{N}$ , where here and in the sequel, we denote  $\mathbb{N} = \{0, 1, 2, 3, \dots\}$  :

1)  $T$  is given by the usual shift  $T(x) = x + 1$ ,  $x \in \mathbb{N}$ .

If  $\Delta \in \mathcal{FP}(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ , then obviously there exists  $A \in \Delta$  such that  $A$  is infinite. Furthermore, for every  $x \in \mathbb{N}$ , we have

$$(1.5) \quad \Delta(x) = \{ A \in \Delta \mid A \text{ is infinite} \} \neq \phi$$

2) If  $T$  is the identity mapping then clearly

$$(1.6) \quad \Delta(x) = \{ A \} \neq \phi, \quad \text{where } x \in A \in \Delta$$

3) Let us now assume that, for a given  $x_* \in \mathbb{N}$ , we have  $T(x) = x_*$ , with  $x \in \mathbb{N}$ . Then obviously

$$(1.7) \quad \Delta(x) = \{ A \} \neq \phi, \quad \text{where } x_* \in A \in \Delta$$

### Remark 1

The above general setup clearly contains as a particular case the following one which is of a wide interest in Ergodic Theory, namely,  $(X, \Sigma, \nu)$ , where  $\Sigma$  is a  $\sigma$ -algebra on  $X$ , while  $\nu$  is a probability on  $(X, \Sigma)$ . In that case, the mapping  $T$  is supposed to satisfy the conditions

$$(1.8) \quad T^{-1}(\Sigma) \subseteq \Sigma$$

and

$$(1.9) \quad \nu(T^{-1}(A)) = \nu(A), \quad A \in \Sigma$$

We note that  $\Sigma$  being a  $\sigma$ -algebra, we have in particular

$$\begin{aligned}
& *) \quad \forall A, A' \in \Sigma : A'' = A \cap A' \in \Sigma \\
(1.10) \quad & **) \quad \phi, X \in \Sigma
\end{aligned}$$

consequently

$$(1.11) \quad \mathcal{FP}(X, \Sigma) \neq \phi$$

## 2. Towards a Solution

First we observe the following natural structure on  $\mathcal{FP}(X, \Sigma)$ , given by the concept of *refinement*. Namely, if  $\Delta, \Delta' \in \mathcal{FP}(X, \Sigma)$ , we define

$$(2.1) \quad \Delta \leq \Delta'$$

if and only if

$$(2.2) \quad \forall A' \in \Delta' : \exists A \in \Delta : A' \subseteq A$$

and in view of that, we can define the mapping

$$(2.3) \quad \psi_{\Delta', \Delta} : \Delta' \longrightarrow \Delta$$

by

$$(2.4) \quad A' \subseteq A = \psi_{\Delta', \Delta}(A'), \quad A' \in \Delta'$$

Then we obtain

### Lemma 1

$$(2.5) \quad \psi_{\Delta', \Delta}(\Delta'(x)) \subseteq \Delta(x), \quad x \in X$$

### Proof

If  $A' \in \Delta'(x)$ , then (1.3) gives

$\{ n \in \mathbb{N}_+ \mid T^n(x) \in A' \}$  is infinite

but in view of (2.4), we have

$$A' \subseteq \psi_{\Delta', \Delta}(A')$$

hence

$\{ n \in \mathbb{N}_+ \mid T^n(x) \in \psi_{\Delta', \Delta}(A') \}$  is infinite

thus (2.5). □

Let us pursue the consequences of the above result in (2.5). In this regard we note that in the usual particular case in Remark 1, the partial order (2.1) on  $\mathcal{FP}(X, \Sigma)$  is in fact *directed*, and obviously has the following stronger property

$$(2.6) \quad \forall \Delta, \Delta' \in \mathcal{FP}(X, \Sigma) : \exists \Delta \vee \Delta' \in \mathcal{FP}(X, \Sigma)$$

since

$$(2.7) \quad \Delta \vee \Delta' = \{ A \cap A' \mid A \in \Delta, A' \in \Delta', A \cap A' \neq \emptyset \}$$

However, for a greater generality, let us consider in Problem 1 not only the whole of  $\mathcal{FP}(X, \Sigma)$ , but also arbitrary subsets of it. Let therefore  $(\Lambda, \leq)$  be any partially ordered set, and consider a mapping

$$(2.8) \quad \Lambda \ni \lambda \longmapsto \Delta_\lambda \in \mathcal{FP}(X, \Sigma)$$

such that

$$(2.9) \quad \lambda \leq \lambda' \implies \Delta_\lambda \leq \Delta_{\lambda'}$$

We call the family  $(\Delta_\lambda)_{\lambda \in \Lambda}$  a *refinement chain*.

Obviously, in view of the above,  $\mathcal{FP}(X, \Sigma)$  itself is such a refinement chain, namely, with  $\Lambda = \mathcal{FP}(X, \Sigma)$ , the partial order in (2.1), and

with the identity mapping in (2.8).

The main point to note is the following. Given  $x \in X$ , then (1.4) implies

$$(2.10) \quad \Delta_\lambda(x) \neq \phi, \quad \lambda \in \Lambda$$

Hence in view of (2.5), (2.9), we have for  $\lambda \leq \lambda'$

$$(2.11) \quad \phi \neq \psi_{\Delta_{\lambda'}, \Delta_\lambda}(\Delta_{\lambda'}(x)) \subseteq \Delta_\lambda(x)$$

Now, based on (2.10), let us use the notation

$$(2.12) \quad \Delta_\lambda(x) = \{ A_{\lambda,1}(x), \dots, A_{\lambda,m_\lambda}(x) \}$$

where  $m_\lambda \geq 1$ , and  $\phi \neq A_{\lambda,j}(x) \in \Sigma$ , with  $1 \leq j \leq m_\lambda$ .

## Problem 2

A more precise reformulation of Problem 1 is as follows. We can investigate whether for a given  $x \in X$ , one or the other of the following two properties may hold, namely

$$(2.13) \quad \exists \Lambda \ni \lambda \longmapsto A_\lambda \in \Delta_\lambda(x) : \bigcap_{\lambda \in \Lambda} A_\lambda \neq \phi$$

or what appears to be a milder property

$$(2.14) \quad \varprojlim_{\lambda \in \Lambda} \Delta_\lambda(x) \neq \phi$$

□

The *inverse limit*, [1, p. 191], in (2.14) is of the family

$$(2.15) \quad (\Delta_\lambda(x) \mid \lambda \in \Lambda)$$

with the mappings, see (2.8), (2.10)

$$(2.16) \quad \psi_{\lambda', \lambda, x} : \Delta_{\lambda'}(x) \longrightarrow \Delta_\lambda(x)$$

for  $\lambda, \lambda' \in \Lambda$ ,  $\lambda \leq \lambda'$  where

$$(2.17) \quad \psi_{\lambda', \lambda} : \Delta_{\lambda'} \longrightarrow \Delta_{\lambda}$$

is given by

$$(2.18) \quad \psi_{\lambda', \lambda} = \psi_{\Delta_{\lambda'}, \Delta_{\lambda}}$$

while

$$(2.19) \quad \psi_{\lambda', \lambda, x} = \psi_{\lambda', \lambda} \mid_{\Delta_{\lambda'}(x)}$$

In order to establish (2.14), we recall the definition of the inverse limit, namely

$$(2.20) \quad \varprojlim_{\lambda \in \Lambda} \Delta_{\lambda}(x) = \left\{ (A_{\lambda} \mid \lambda \in \Lambda) \in \prod_{\lambda \in \Lambda} \Delta_{\lambda}(x) \mid \begin{array}{l} \forall \lambda, \lambda' \in \Lambda, \lambda \leq \lambda' : \\ \psi_{\lambda', \lambda, x}(A_{\lambda'}) = A_{\lambda} \end{array} \right\}$$

We note that in the definition of the inverse limit, the partial order  $\leq$  on  $\Lambda$  can be arbitrary, and in fact, it can be a mere pre-order.

Further we note, [1, Exercise 4, no. 4, p. 252], that an inverse limit such as for instance in (2.20), can be void even when all sets  $\Delta_{\lambda}(x)$  are nonvoid and all mappings  $\psi_{\lambda', \lambda, x}$  are surjective.

However, as seen in Theorem 1 in the sequel, this is not the case in (2.14).

Meanwhile, for the sake of further clarification, we consider (2.20) in the following particular case.

## Example 2

In the case 1) of Example 1, let us consider  $(\Lambda, \leq) = \mathbb{N}$ , and take the following sequence of finite partitions of  $\mathbb{N}$

$$(2.21) \quad \mathbb{N} \ni \lambda \longmapsto \Delta_\lambda \in \mathcal{FP}(\mathbb{N}, \mathcal{P}(\mathbb{N}))$$

where

$$(2.22) \quad \begin{aligned} \Delta_0 &= \{ \mathbb{N} \} \\ \Delta_1 &= \{ \{0\}, \{1, 2, 3, \dots\} \} \\ \Delta_2 &= \{ \{0\}, \{1\}, \{2, 3, 4, \dots\} \} \\ \Delta_3 &= \{ \{0\}, \{1\}, \{2\}, \{3, 4, 5, \dots\} \} \\ &\dots\dots\dots \end{aligned}$$

thus clearly  $\Delta_0 \leq \Delta_1 \leq \Delta_2 \leq \Delta_3 \leq \dots$

Now if we take  $x = 0 \in \mathbb{N} = X$ , then, see (2.12)

$$(2.23) \quad \Delta_\lambda(x) = \{ A_{\lambda,x} = \{\lambda, \lambda + 1, \lambda + 2, \dots\} \}, \quad \lambda \in \Lambda$$

Therefore (2.13) *fails* to hold, since obviously

$$(2.24) \quad \bigcap_{\lambda \in \Lambda} A_{\lambda,x} = \phi$$

On the other hand, regarding (2.14), in view of (2.20), (2.23), as well as (2.16) - (2.19), we obtain

$$(2.25) \quad \varprojlim_{\lambda \in \Lambda} \Delta_\lambda(x) = \{ (A_{\lambda,x} \mid \lambda \in \Lambda) \} \neq \phi$$

In the case 2) of Example 1, for  $x \in \mathbb{N}$ , we have, see (1.6)

$$(2.26) \quad \Delta_\lambda(x) = \{ A_{\lambda,x} \}$$

where

$$(2.27) \quad A_{\lambda,x} = \begin{cases} \{x\} & \text{if } x < \lambda \\ \{\lambda, \lambda + 1, \lambda + 2, \dots\} & \text{if } x \geq \lambda \end{cases}$$

thus (2.13) will this time hold, since (2.26), (2.27) obviously yield for  $x \in \mathbb{N}$

$$(2.28) \quad \bigcap_{\lambda \in \Lambda} A_{\lambda,x} = \{x\} \neq \phi$$



As for (2.14), the relations (2.26), (2.27) applied to (2.20) give

$$(2.29) \quad \varprojlim_{\lambda \in \Lambda} \Delta_\lambda(x) = \{ (A_{\lambda,x} \mid \lambda \in \Lambda) \} \neq \phi$$

which in view of (2.26) - (2.28) means essentially that

$$(2.30) \quad \varprojlim_{\lambda \in \Lambda} \Delta_\lambda(x) = \{ \{x\} \} \neq \phi$$

Lastly, in the case 3) of Example 1, we have, see (1.7)

$$(2.31) \quad \Delta_\lambda(x) = \{ A_{\lambda,x} \}, \quad x \in \mathbb{N}$$

where

$$(2.32) \quad A_{\lambda,x} = \begin{cases} \{x_*\} & \text{if } x_* < \lambda \\ \{\lambda, \lambda+1, \lambda+2, \dots\} & \text{if } x_* \geq \lambda \end{cases}$$

hence (2.13) holds again, since

$$(2.33) \quad \bigcap_{\lambda \in \Lambda} A_{\lambda,x} = \{x_*\} \neq \phi$$

while (2.14) takes the form

$$(2.34) \quad \varprojlim_{\lambda \in \Lambda} \Delta_\lambda(x) = \{ (A_{\lambda,x} \mid \lambda \in \Lambda) \} \neq \phi$$

which in view of (2.32) means essentially that

$$(2.35) \quad \varprojlim_{\lambda \in \Lambda} \Delta_\lambda(x) = \{ \{x_*\} \} \neq \phi$$

## Remark 2

The three instances in Example 2 above, with their respective versions (2.25), (2.29), (2.30), (2.34) and (2.35) of problem (2.14) as formulated in Problem 2, can give a motivation for the use of the *inverse limits* in Ergodic Theory. Indeed, in each of these three cases, the corresponding inverse limits reflect in a nontrivial manner obvious ergodic properties of the specific mappings  $T$  involved.

In this regard, the relevance of the inverse limit is particularly clear

in the first instance in Example 2, namely, when  $T : \mathbb{N} \longrightarrow \mathbb{N}$  is the usual shift, and when problem (2.13), as formulated in Problem 2, has a solution in (2.24) which does not give much information about  $T$ , since the same relation may be obtained for many other mappings of  $\mathbb{N}$  into itself.

On the other hand, the inverse limit in (2.25) does give an information which is clearly more revealing about the specific feature of  $T$ .

Of course, in analyzing the ergodic features of mappings  $T$  of  $\mathbb{N}$  into itself, one can use a variety of other *refinement chains*, than the particular one in (2.21), (2.22). We consider next such an example of a different refinement chain in the case 1) of Example 1.

### Example 3

Let  $X = \mathbb{N}$ ,  $\Sigma = \mathcal{P}(\mathbb{N})$  and consider the mapping  $T : \mathbb{N} \longrightarrow \mathbb{N}$  given by the usual shift  $T(x) = x + 1$ ,  $x \in \mathbb{N}$ .

Let  $\mathcal{U}$  be a *free ultrafilter* on  $X = \mathbb{N}$  which, we recall, means a filter with the following two properties

$$(2.36) \quad \forall A \subseteq \mathbb{N} : \text{either } A \in \mathcal{U}, \text{ or } \mathbb{N} \setminus A \in \mathcal{U}$$

$$(2.37) \quad \bigcap_{U \in \mathcal{U}} U = \emptyset$$

These two conditions imply that

$$(2.38) \quad \forall U \in \mathcal{U} : U \text{ is infinite}$$

Furthermore, we also have that

$$(2.39) \quad \exists U \in \mathcal{U} : \mathbb{N} \setminus U \text{ is infinite}$$

For  $U \in \mathcal{U}$ , let us consider the set of finite partitions  $\Delta$  of  $\mathbb{N}$  which contain  $U$ , that is, given by

$$(2.40) \quad \mathcal{FP}_U(\mathbb{N}, \mathcal{P}(\mathbb{N})) = \{ \Delta \in \mathcal{FP}(\mathbb{N}, \mathcal{P}(\mathbb{N})) \mid U \in \Delta \}$$

Further, let us consider

$$\begin{aligned}
(2.41) \quad \mathcal{FP}_{\mathcal{U}}(\mathbb{N}, \mathcal{P}(\mathbb{N})) &= \bigcup_{U \in \mathcal{U}} \mathcal{FP}_U(\mathbb{N}, \mathcal{P}(\mathbb{N})) = \\
&= \{ \Delta \in \mathcal{FP}(\mathbb{N}, \mathcal{P}(\mathbb{N})) \mid \exists U \in \mathcal{U} : U \in \Delta \}
\end{aligned}$$

We shall take now  $(\Lambda, \leq) = \mathcal{FP}_{\mathcal{U}}(\mathbb{N}, \mathcal{P}(\mathbb{N}))$  endowed with the partial order  $\leq$  in (2.1) which corresponds to the usual refinement of partitions. Finally, the mapping (2.8), (2.9) will simply be the identity mapping

$$(2.42) \quad \Lambda \ni \lambda = \Delta \longmapsto \Delta \in \mathcal{FP}_{\mathcal{U}}(\mathbb{N}, \mathcal{P}(\mathbb{N}))$$

Given  $U \in \mathcal{U}$ ,  $\Delta \in \mathcal{FP}_{\mathcal{U}}(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ , with  $U \in \Delta$ , as well as  $x \in \mathbb{N}$ , it follows easily that

$$(2.43) \quad U \in \Delta(x)$$

and in fact, we have the stronger property, similar with (1.5), namely

$$(2.44) \quad \Delta(x) = \{ A \in \Delta \mid A \text{ is infinite} \}$$

Now it is easy to see that, in view of (2.20), we obtain

$$(2.45) \quad (A_{\lambda} \mid \lambda \in \Lambda) \in \varprojlim_{\lambda \in \Lambda} \Delta_{\lambda}(x)$$

where for  $\lambda = \Delta \in \mathcal{FP}_{\mathcal{U}}(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ , we have

$$(2.46) \quad \Delta_{\lambda} = \Delta, \quad A_{\lambda} = U$$

hence

$$(2.47) \quad \varprojlim_{\lambda \in \Lambda} \Delta_{\lambda}(x) \neq \phi$$

### 3. A General Inverse Limit Ergodic Result

As seen in the theorem next, the result in (2.25) is in fact a particular case of a rather general one.

### Theorem 1

Let  $(X, \Sigma, T)$  be as at the beginning of section 1. Further, let  $(\Lambda, \leq)$  be a *directed* partial order, together with a mapping, see (2.8)

$$\Lambda \ni \lambda \longmapsto \Delta_\lambda \in \mathcal{FP}(X, \Sigma)$$

which satisfies (2.9), as well as the following condition :

$$(3.1) \quad \exists \Lambda_0 \subseteq \Lambda : \Lambda_0 \text{ is } \textit{countable} \text{ and } \textit{cofinal} \text{ in } \Lambda$$

Then for every  $x \in X$ , we have

$$(3.2) \quad \varprojlim_{\lambda \in \Lambda} \Delta_\lambda(x) \neq \phi$$

### Proof.

It follows from Proposition 5 in [1, p. 198], whose conditions are satisfied, as shown next.

Indeed, given  $x \in X$ , in view of (2.10), we have

$$(3.3) \quad \Delta_\lambda(x) \neq \phi, \quad \lambda \in \Lambda$$

Further, (2.16) gives for  $\lambda, \lambda' \in \Lambda$ ,  $\lambda \leq \lambda'$  the mapping

$$(3.4) \quad \psi_{\lambda', \lambda, x} : \Delta_{\lambda'} \longrightarrow \Delta_\lambda(x)$$

and obviously, see (2.17) - (2.19), (2.3), (2.4)

$$(3.5) \quad \psi_{\lambda, \lambda, x} = id_{\Delta_\lambda(x)}$$

while for  $\lambda, \lambda', \lambda'' \in \Lambda$ ,  $\lambda \leq \lambda' \leq \lambda''$  we have

$$(3.6) \quad \psi_{\lambda', \lambda, x} \circ \psi_{\lambda'', \lambda', x} = \psi_{\lambda'', \lambda, x}$$

Lastly, the mappings (3.3) are surjective. Indeed, let  $A \in \Delta_\lambda(x)$ , then we have to find  $A' \in \Delta_{\lambda'}(x)$ , such that

$$(3.7) \quad \psi_{\lambda', \lambda, x}(A') = A$$

But (1.3) yields

$$(3.8) \quad \{ n \in \mathbb{N}_+ \mid T^n(x) \in A \} \text{ is infinite}$$

Therefore (2.1) - (2.4) will give  $A' \in \Delta_{\lambda'}(x)$ , such that  $A' \subseteq A$ , which means precisely (3.7). □

### Remark 3

1) An important fact in Theorem 1 above is that there are *no* conditions whatsoever required on the mappings  $T : X \longrightarrow X$ .

2) In general, when  $\Sigma$  is uncountable - a case which is often of interest in applications - the set  $\mathcal{FP}(X, \Sigma)$  of all finite partitions of  $X$  with subsets in  $\Sigma$ , see (1.1), will also be uncountable. Furthermore, when considered with the natural partial order (2.1), (2.2), the set  $\mathcal{FP}(X, \Sigma)$  does *not* have a countable cofinal subset. Therefore, in such a case one *cannot* take in Theorem 1

$$(3.9) \quad (\Lambda, \leq) = (\mathcal{FP}(X, \Sigma), \leq)$$

as the directed partial order, and instead, one has to limit oneself to smaller directed partial orders  $(\Lambda, \leq)$ , namely, to those which satisfy condition (3.1).

3) The set  $\Sigma$  can be uncountable even when  $X$  is countable, since one can take, for instance,  $\Sigma = \mathcal{P}(X)$ , that is, the set of all subsets of  $X$ .

## References

- [1] Bourbaki N : Elements of Mathematics, Theory of Sets. Springer, 2004